The Mathematics of Gambling
with Related Applications

Madhu Advani
Stanford University
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“Gambling: the sure way of getting nothing for something”
-Wilson Mizner

“No wife can endure a gambling husband unless he is a steady winner”
-Thomas Dewar
Motivation and Goals

- Accumulate information to give you an edge
- Use that information to the best of your ability
- Be successful in an uncertain environment
1. Gaining an Edge
2. Avoiding Gambler’s ruin
3. Bigger Picture
Section 1

Gaining an Edge
Claude Shannon

Creator of Information theory

“I visualize a time when we will be to robots what dogs are to humans, and I’m rooting for the machines”
Theseus the maze-solving mouse (Machine Learning)
Edward O. Thorp

Father of the wearable computer

Madhu Advani (Stanford University) Mathematics of Gambling

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Why Don’t Casinos change the Rules?

1. They tried
2. Most card counters are bad
3. Knockout dealers
4. You may be banned
5. Or worse
Approximate the quadrant the ball will land in. Bets placed for a second or two after wheel is spun.

Built in Shannon’s basement and kept a secret Edward Thorp now known as the father of the wearable computer.
Coin Flip Game

Fair coin: 50-50 chance heads or tails

For some reason I am giving amazing odds of 3 dollars for every 1 dollar if the coin lands on heads

Imagine you have a dollar and want to play the game: how much would you bet?
Coin Flip Game

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Imagine you have a dollar and want to play the game: how much would you bet?

How about 10 dollars?
Bet all your Money

\[ E \left[ W^{n+1} \right] = \frac{3}{2} W^n \]

The problem is gambler’s ruin: the coin eventually lands on tails
How to Invest

Bet all your Money

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The problem is gambler’s ruin: the coin eventually lands on tails

Long-term investment growth

\[ X_n = X_0 g_1 g_2 \ldots g_n \]

\[ X_n = X_0 2^{\log(g_1) + \log(g_2) + \ldots + \log(g_n)} \]

Central limit theorem says sum independent random \( \approx \) mean:

\[ R_{\text{max}} = \max_b E \left[ \log g \right] \]
Optimal Betting

Maximize expected log of growth!

Say we bet a fraction of our money $b \in [0, 1]$ on heads

$$\log g = \log(Ob + (1 - b)) \quad \text{Pr} = p$$
$$\log g = \log(1 - b) \quad \text{Pr} = 1 - p$$

Odds I’m giving are 3 for 1 so $O = 3$, $p = 1/2$

$$\max_b E[\log g] = \max_b [p \log(Ob + (1 - b)) + (1 - p) \log(1 - b)]$$
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$$\max_{b} E[\log g] = \max_{b} \left[ p \log(Ob + (1 - b)) + (1 - p) \log(1 - b) \right]$$

$$b^* = \frac{p - \frac{1}{O}}{1 - \frac{1}{O}}$$
In 500 games, you become a billionaire with the optimal strategy, a millionaire with the low risk strategy. Interestingly, even with the odds in your favor, you lose all your money if you bet too aggressively.

Comparing fractional betting - Fair coin with 3:1 odds
Optimal Betting for Blackjack

\( O = 1/2 \): Dealers give you 2 for 1 if you win (usually)

If the deck has a high count you have an edge: \( p > 1/2 \)

\[ b = 2p - 1 \]

For a moderately high count \( p = .51 \) so bet 2 percent of your money
Section 3

Bigger Picture
A beautiful theory relating information theory to gambling. Imagine a horse-race with \( n \) horses and odds \( o_i \). If the true probability of each horse winning is \( p_i \). Say you bet a fraction \( b_i \) of your money on each horse, then

\[
R = E[\log(g)] = \sum_i p_i \log(b_i o_i)
\]

\( b_i^* = p_i \)

\[
D(p, q) = \sum_i p_i \log(p_i / q_i)
\]

\( D \) measures the difference between two probability distributions.

Money grows as \( 2^{mD(p, q)} \), if you bet in \( m \) races! \( q_i = \frac{1}{o_i} \). Discrepancy between reality and dealer’s beliefs can be exploited.
Diversify your bets: both hedging against disaster and maximizing growth rate. (Hedging)

Bet your beliefs

Example: Lottery, Securities
Intuition: The amount of predictability in a message tells you how much you can compress it. The higher the entropy of a language the less predictable.

\[ H = \sum_i -p_i \log p_i \]

**Example**

Imagine you want to make a language more efficient than English, but with 2 letter.

Take a dictionary and make a conversion chart

More probably words should have smaller lengths
Intuition: Noise is corrupting your messages. Therefore you may have to repeat your messages or add error correcting bits. How much longer do the messages need to get?

\[ R \leq I(X, Y) = D(p(x, y) \| p(x)p(y)) \]

X is input and Y is corrupted output. You are limited by how dependent the corrupted and original signal are on each other. If they are independent, you can’t send messages!
Conclusions and Wrapping up

You need to find an edge to make money (bet probably not in Vegas)
Even with an edge, you have to bet appropriately to make money

Made a 28 percent return on his portfolio: more than Warren Buffet.
Continued Tinkering: Chess, Rubik’s cube, etc.

Made an average of 20 percent return on investments over 25 years.
Hedge Fund Manager - Currently President of Edward O. Thorp and Associates
References

Fortune’s Formula
THE UNTOLD STORY OF THE SCIENTIFIC BETTING SYSTEM THAT BEAT THE CASINOS AND WALL STREET
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Elements of Information Theory
SECOND EDITION
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